CONTRIBUTION TO THE STUDY OF THE AVERAGE AND FRACTIONAL BIREFRINGENCE OF CIRCULAR HOMOGENEOUS FIBRES

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The birefringent properties of a homogeneous circular fibre between two polarizers are discussed depending on the mutual orientation of all three elements. A simple approximative method is suggested allowing the determination of fibre birefringence from the relative position of the last integer isochromatic fringe, as well as a method of the determination of fractional birefringence in the centre of the fibre from measurements of the intensities ratio for the crossed and parallel position of the polarizer and analyzer.

In anisotropic materials molecules are arranged in a certain preferential direction and the refractive index depends on the direction of propagation of the light wave. In a general case the anisotropic material has three principal refractive indices, but fibres exhibit cylindrical symmetry, so that two refractive indices are sufficient for their characterization, namely $n_{\parallel}$ for light polarized parallel to the fibre direction and $n_{\perp}$ for light polarized perpendicularly to the same direction. The difference $\Delta n = n_{\parallel} - n_{\perp}$ between the two refractive indices is called birefringence. The determination of Δn usually involves measurement of optical retardation $\delta = (2\pi/\lambda)\,\Delta nd$ and of the fibre diameter d. As has been demonstrated[1], the use of phenomenological description is effective for the investigation of birefringence; in this case the birefringent medium is represented by Mueller's matrix of a linear retarder (the fibre causes retardation of one component owing to different velocities of light propagation in the fibre direction and perpendicular to it). Depending on the magnitude of birefringence and the required accuracy of measurement one chooses various methods of measurement[2].

The paper presents an analysis of the behaviour of a birefringent homogeneous circular fibre between two polarizers (situations arising during observations in a polarizing microscope) and suggests the determination of a fractional retardation in the centre of the fibre.

THEORETICAL

Let ε be an angle between the direction of the fast axis of a linear retarder (direction of the lower refractive index) and the horizontal axis (Fig. 1). The optical behaviour of the fibre is then determined[1] by the matrix $\mathbf{W}$,

$$\mathbf{W}(\varepsilon, \delta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2^2 + s_2^2 \cos \delta & c_2 s_2(1 - \cos \delta) & -s_2 \sin \delta \\ 0 & c_2 s_2(1 - \cos \delta) & s_2^2 + c_2^2 \cos \delta & c_2 \sin \delta \\ 0 & s_2 \sin \delta & -c_2 \sin \delta & \cos \delta \end{pmatrix}, \qquad (1)$$

where $c_2 = \cos 2\varepsilon$, $s_2 = \sin 2\varepsilon$, $\delta = (2\pi/\lambda)\, \Delta nd$, $\Delta n = n_{\parallel} - n_{\perp}$, d is the thickness of material, λ is the wave-length of light. Symbolically, the optical behaviour of a birefringent material between two polarizers in a general position is described by[1]

$$\boldsymbol{S}' = \mathbf{A}(\beta)\ \mathbf{W}(\varepsilon, \delta)\, \mathbf{P}(\alpha)\, \boldsymbol{S}\,, \qquad (2)$$

where $\boldsymbol{S}, \boldsymbol{S}'$ are Stokes' vectors of the incident and emerging radiation[3], $\boldsymbol{S} = (\boldsymbol{I}, \boldsymbol{M}, \boldsymbol{C}, \boldsymbol{S})^{\mathrm{T}}$, $\boldsymbol{S}' = (\boldsymbol{I}', \boldsymbol{M}', \boldsymbol{C}', \boldsymbol{S}')^{\mathrm{T}}$, defined in terms of the mean values from the horizontal a_x and vertical, a_y, component of the field vector and of the phase angle γ between them. If the time mean value is denoted by angular brackets, it holds

$$\boldsymbol{I} = \langle a_x^2 + a_y^2 \rangle\,, \quad \boldsymbol{M} = \langle a_x^2 - a_y^2 \rangle\,, \quad \boldsymbol{C} = \langle 2a_x a_y \cos \gamma \rangle\,,$$
$$\boldsymbol{S} = \langle 2a_x a_y \sin \gamma \rangle\,. \qquad (3)$$

$\mathbf{P}(\alpha)$, $\mathbf{A}(\beta)$ are Mueller's matrices of the linear polarizer with an angle of orientation α and of the analyzer with an angle of orientation β. It holds

$$\mathbf{P}(\alpha) = (1/2) \begin{pmatrix} 1 & \boldsymbol{C}_2 & \boldsymbol{S}_2 & 0 \\ \boldsymbol{C}_2 & \boldsymbol{C}_2^2 & \boldsymbol{C}_2\boldsymbol{S}_2 & 0 \\ \boldsymbol{S}_2 & \boldsymbol{C}_2\boldsymbol{S}_2 & \boldsymbol{S}_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (4)$$

where $\boldsymbol{C}_2 = \cos 2\alpha$, $\boldsymbol{S}_2 = \sin 2\alpha$; a similar relationship holds for $\mathbf{A}(\beta)$. If the usual experimental arrangement is used, $\alpha = 0$, $\beta = 90^0$ (crossed polarizers); $I'_{\perp}$ is the respective intensity. For the intensity $I'_{\perp}$ we obtain from relationship (2) after simple rearrangements (first component of the vector $\boldsymbol{S}'$)

$$I'_{\perp} = (I_0/2) \sin^2(\delta/2)\, s_2^2\,. \qquad (5)$$

Similarly, substitution into (2) gives for the intensity $I'_{\parallel}$ (parallel position of the polarizer and analyzer, $\alpha = \beta = 0$)

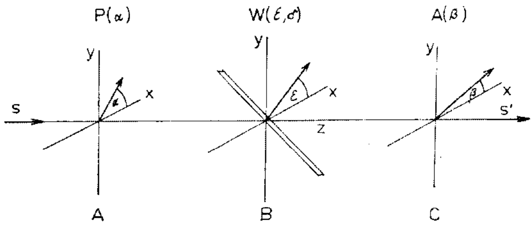

FIG. 1

Scheme of Experimental Arrangement

A Polarizer **P**(α) with an angle of orientation α, B fibre **W** (ε, δ) having the fast axis at an angle ε with the horizontal axis, C analyzer **A**(β) having an orientation angle β.

$$I'_{\parallel} = (I_0/2)\left(1 - s_2^2 \sin^2 \delta/2\right). \tag{6}$$

The comparison of (5) and (6) gives the known complementarity of intensities[4]

$$I'_{\parallel} = I_0 - I'_{\perp}. \tag{7}$$

By writing (5) in the form

$$I'_{\perp} = I_0 \, F_1(\delta) \, F_2(\varepsilon) \tag{8}$$

we obtain a formally separated part corresponding to the isoclinic lines $F_2(\varepsilon) = 0$ and a part corresponding to the isochromatic fringes $F_1(\delta) = 0$. In the usual experimental arrangement we choose $\varepsilon = 45°$, which allows the maximum of the function $F_2(\varepsilon)$ to be obtained. A system of isochromatic fringes parallel to the fibre axis is then observed against a dark background (the polarization microscope in the position of crossed polarizers, monochromatic light), fringes becoming more concentrated at the edge of the fibre[4] (Fig. 2). If the fibre is assumed to have a circular cross-section, a variable y can be introduced which characterizes the position of the fringes in the field of view and is related by a simple equation

$$d = 2(R^2 - y^2)^{1/2} \tag{9}$$

with the fibre radius R and a thickness d where the retardation occurs. Eq. (5) can then be written as

$$I'_{\perp}(y) = (I_0/2)\sin^2\left[(2\pi/\lambda)\,\Delta n(R^2 - y^2)^{1/2}\right] s_2^2 . \tag{10}$$

For a position of the fibre with the axis bisecting the angle between the crossed polarizers ($\varepsilon = 45°$) it holds

$$I'_\perp(y) = (I_0/2)\sin^2\left[(2\pi/\lambda)\,\Delta n(R^2 - y^2)^{1/2}\right]. \tag{11}$$

By introducing relative quantity $\varrho = y/R$ we obtain

$$I'_\perp(\varrho) = (I_0/2)\sin^2\left[(2\pi/\lambda)\,\Delta n\,R(1 - \varrho^2)^{1/2}\right]. \tag{12}$$

From the condition for the extremum $\partial I'_\perp/\partial\varrho = 0$ we obtain for the relative position of maxima

$$\varrho_{\mathrm{M}} = (1 - N^2\lambda^2/16\,\Delta n^2R^2)^{1/2}\,, \quad N = 0, 1, 2, \ldots \tag{13}$$

Similarly, for the zero points of the function $I'_\perp(\varrho)$, *i.e.* for the position of isochromatic fringes we obtain the condition

$$\varrho_{\mathrm{m}} = (1 - N^2\lambda^2/4\,\Delta n^2R^2)^{1/2}\,, \quad N = 0, 1, 2, \ldots \tag{14}$$

If the number of the orders N is unknown, then for adjacent fringes it holds, assuming constant birefringence[4],

$$\begin{aligned}\Delta n d_2 &= (N - 1)\,\lambda\,,\\ \Delta n d_1 &= N\lambda\,,\end{aligned} \tag{15}$$

where d_1, d_2 are lengths of secants of the circular fibre. Hence for the number of the

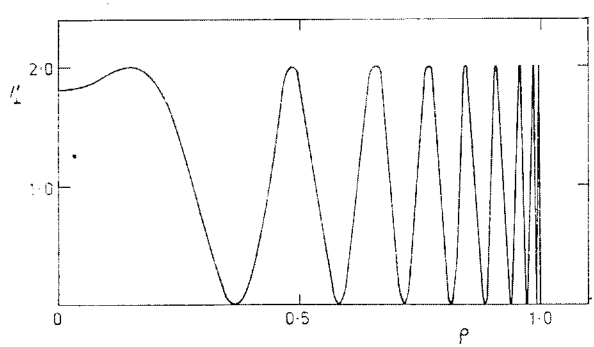

FIG. 2

Light Intensity $I'_\perp(\varrho)$ (in relative units) at Crossed Polarizer and Analyzer for Fibre Parameters: $\Delta n = 6 \cdot 10^{-2}$, $R = 43\ \mu$, $\lambda = 0{\cdot}6\ \mu$.

orders N it holds

$$N = d_1/(d_1 - d_2) = (R^2 - x_1^2)^{1/2}/[(R^2 - x_1^2)^{1/2} - (R^2 - x_2^2)^{1/2}], \quad (16)$$

where x_1 and x_2 are distances between the centres of adjacent fringes. Usually a region remains around the centre of the fibre where the intensity does not vanish completely, *i.e.* the last integer fringe does not lie in the centre of the fibre. To determine the total birefringence it is necessary to determine the fractional birefringence in the centre of the fibre which corresponds to the retardation of the fraction ΔN. There exists however a possibility to measure the average birefringence without knowing the fractional birefringence. Let us measure the relative positions of the last order of the integer fringe ϱ_N which is obtained by determining the distance between the centres of the last fringes symmetrical over the fibre centre (Fig. 2). This distance divided by the fibre diameter gives ϱ_N. From (*14*) we determine Δn for a known N and ϱ_N without knowing the value of the fractional birefringence in the fibre centre. If we are interested in the latter, a similar procedure may be employed. From the known position of the last fringe ϱ_N we determine the fictitious position of the following order by using the relationship

$$\varrho_f = |[1 - (N + 1)^2 \lambda^2/4 \Delta n^2 R^2]|^{1/2}; \quad (14a)$$

the ratio $\Delta N = \varrho_N/(\varrho_f + \varrho_N)$ gives the fraction ΔN. The determination of the fractional birefringence (the non-integer order of fringe) is illustrated by an example (*cf.* Appendix). Besides the possibility of determination of the fractional birefringence by employing the extinction angle or other methods[2], the intensity method can also be used. As has been shown[1], it holds for the retardation

$$\cos \delta = (I'_{\parallel} - I'_{\perp})/(I'_{\parallel} + I'_{\perp}). \quad (17)$$

Theoretically, the above relationship could be used for the determination of the course of retardation over the whole cross-section of the fibre, because it can also be applied to the wedge-shaped section of the fibre which is used for a precise reading of the number of fringes[4]. If a polarizing microscope with a photographic apparatus or photoelectric detection is employed, the above relationship can be used at least for the determination of the fractional birefringence in the centre of the fibre. An example is given in the Appendix.

DISCUSSION

Eq. (*12*) determines the course of the intensity of light passed through a homogeneous circular fibre placed so that the fibre axis bisects the angle between crossed polarizers depending on the relative position ϱ along the fibre radius. $\varrho = 0$ corresponds to the centre of the fibre and $\varrho = 1$ corresponds to its edge. If the fibre is placed at a different angle ε, expression (*12*) must include a multiplicative term corresponding to isoclinic lines and having the explicit form $\sin^2 2\varepsilon$. The relationship for the intensity $I'_{\parallel}$ is obtained by means of complementarity (*7*). For a general position of the polarizer, fibre and analyzer Eq. (*2*) must be employed and the position of the individual elements must be specified. Eq. (*5*) holds also for a nonhomogeneous fibre having a general cross-section, so that Eq. (*17*) ensuing therefrom can be used in principle for the determination of the course of birefringence in the fibre, but the experimental technique existing so far does not make it possible. Eq. (*5*) can also be used for a wedge-shaped fibre cross-section employed for the precise reading of the number of fringes[4]. The meaning of Eq. (*16*) is greatly limited by the assumption of validity (homogeneous fibre having a circular cross-section); the result is also strongly influenced by the accuracy with which the two quantities, x_1 and x_2, have been determined. It could have some meaning for a homogeneous fibre having a high number of fringe orders with a possible use of the iterative procedure. The suggested method of determination of the linear average birefringence is an approximative one and should be applied to real fibres with caution. However, even nonhomogeneous fibres exhibit as a rule the smallest change in Δn with the relative position near the fibre centre[4,5], which partly justified the use of this method also for nonhomogeneous fibres.

APPENDIX

Let us illustrate the relationships derived in this paper by a fibre having the following parameters: $\Delta n = 6 \,.\, 10^{-2}$, $R = 43\,\mu$, $\lambda = 0{\cdot}6\,\mu$. The course of intensity $I'_{\perp}$ depending on the relative position according to Eq. (*12*) is shown in Fig. 2. By using the relative position of the fringe of the 8th order ($\varrho_8 \approx 0{\cdot}37$) we shall determine the average birefringence by means of Eq. (*14*) without knowing the fractional birefringence. If we want to know the fractional birefringence, we shall determine from (*17*) the fictitious position of the 9th order $\varrho_9 \approx 0{\cdot}31$. The ratio $0{\cdot}37/(0{\cdot}31 + 0{\cdot}37)$ gives an approximate value of the fractional order of the fringe $N \approx 0{\cdot}54$. The average birefringence of the fibre is then

$$\Delta n = (N + \Delta N)/d \approx (8 + 0{\cdot}54)\, 0{\cdot}6/2{\cdot}43 = 0{\cdot}0596\,.$$

By using Eq. (*17*) we obtain for the fractional birefringence

$$\cos \delta = (I'_{\parallel}(0) - I'_{\perp}(0)/(I'_{\parallel}(0) + I'_{\perp}(0)) = (0{\cdot}191 - 1{\cdot}809)/2 = -0{\cdot}809\,,$$

which corresponds to $\Delta N = 0{\cdot}6$. The average birefringence then is $\Delta n = (8 + 0{\cdot}6)\,.\,0{\cdot}6/2{\cdot}43 = 0{\cdot}06$. Thus, the method used for an approximative determination of the fractional order introduces into the determination of birefringence an error lower than 1%. Obviously, the error is much smaller than for instance an incorrect determination of orders.

REFERENCES

1. Holoubek J.: Czech. J. Phys. B *24*, 1162 (1974).
2. *Physical Methods of Investigating Textiles* (R. Meredith, J. W. S. Hearle, Eds). Textile Book Publ., New York 1959.
3. Shurcliff W. A.: *Polarized Light.* Harvard Univ. Press, Cambridge (Mass.) 1962.
4. Andrews D. R.: J. Appl. Phys. *25*, 1223 (1954).
5. Hannes H.: Kolloid-Z. Z. Polym. *250*, 765 (1972).

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